Distortion Maps for Elliptic Curves over Finite Fields

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Initially a CIMPA school project

CIMPA stands for: Centre International de Mathématiques Pures et Appliquées.

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Distorting a Point

Elliptic Curves over Finite Fields

Let $\mathbb{F}_q$ be a finite field, where q is a power of a prime number p. An Elliptic Curve E over $\mathbb{F}_q$ is a non-singular projective curve of genus 1 with a specified $\mathbb{F}_q$-rational point $\mathcal{O}$. If $p \neq 2, 3$, it is given by an equation:

$$E/\mathbb{F}_q : \; y^2 = x^3 + Ax + B, \quad 4A^3 + 27B^2 \neq 0, \quad A, B \in \mathbb{F}_q$$

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By $\mathrm{End}(E)$ we denote the ring of (geometric) endomorphisms of E given by rational functions with coefficients in an algebraic closure $\overline{\mathbb{F}}_q$, which respect the group structure of $E(\overline{\mathbb{F}}_q)$.

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E.g. $E/\mathbb{F}_5 : \; y^2 = x^3 + 1$ with the map $(x, y) \mapsto (\zeta \cdot x, y)$, where ζ is a primitive cube root of unity (defined over $\mathbb{F}_{5^2}$!).

Distorting a Point

Definition: Distortion map

Let P in $E(\mathbb{F}_q)$. A *distortion map* ϕ with respect to P is an element in $\mathrm{End}(E)$ such that $\phi(P) \notin \langle P \rangle$, i.e. it maps the point P outside the cyclic group P generates.

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Distorting m-torsion points

Denote by $E[m] = \{P \in E(\overline{\mathbb{F}}_q) : \ m \cdot P = \mathcal{O}\}$ the m-torsion points of $E(\overline{\mathbb{F}}_q)$. We say ϕ is a distortion map for $E[m]$ if it is a distortion map for every P in $E[m]$.

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The Weil Pairing

The group $E[m]$ comes equipped with a bilinear, non-degenerate pairing called the Weil pairing: $e_m : E[m] \times E[m] \longrightarrow \mu_m$, where $e_m(P, Q)$ is an m-th root of unity. In particular, $e_m(P, P) = 1$ for every m-torsion point P.

A modification

Modified Weil pairing

Given a distortion map ϕ with respect to $E[m]$, we define the modified Weil pairing

$$e_{m,\phi} : E[m] \times E[m] \to \mu_m$$

$$e_{m,\phi}(P, Q) = e_m(P, \phi(Q)).$$

In particular, $e_{m,\phi}(P, P)$ is an m-th root of unity $\neq 1$.

Some applications

Discrete Logarithm Problem: Knowing P and $a \cdot P$, find a.

A toy-example: Tripartite secret key

Suppose three parties Alice, Bob and Chris want to agree on a shared secret based on the public parameters $(E, \mathbb{F}_q, P)$ where P is an m-torsion point and $E[m]$ admits a known distortion map ϕ. Each of them has private keys a, b, c and public keys $a \cdot P$, $b \cdot P$, $c \cdot P$ respectively.

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Each one of them has enough information to compute:

$$e_{m,\phi}(P,P)^{abc} = e_{m,\phi}(b \cdot P, c \cdot P)^a = e_{m,\phi}(a \cdot P, c \cdot P)^b = e_{m,\phi}(a \cdot P, b \cdot P)^c,$$

and use it as a key!

Attack vs. Defense

Breaking Decision Diffie-Hellman (DDH)

Because $e_{m,\phi}(P,P) \neq 1$, we can easily distinguish triples (aP, bP, cP) from $(aP, bP, (ab)P)$ by checking if they pair correctly:

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The MOV Attack & Verheul's Theorem

- **MOV Attack:** The map $P \longmapsto e_m(P,Q)$ embeds the curve into $\mathbb{F}^*_{q^k}$. This reduces the hard Discrete Logarithm Problem on the curve to an easier problem in the finite field!
- **Verheul's Theorem:** If this MOV embedding has a computable inverse, we can easily compute $(ab) \cdot P$ from $a \cdot P$ and $b \cdot P$. Recall, the Elliptic Curve Diffie-Helmann key exchange provides the key $(ab) \cdot P = a \cdot (b \cdot P) = b \cdot (a \cdot P)$.

A quote (or not)

As Hilbert famously said (not really)

We must know about the existence of distortion maps for elliptic curves over finite fields.

We will know!

Back to Endomorphisms

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In this case an elliptic curve is a torus $\mathbb{C}/\Lambda$ for a $\mathbb{Z}$-lattice $\Lambda = \langle \omega_1, \omega_2 \rangle$.

$$\mathrm{End}(E) = \mathbb{Z} \text{ (usually)}$$

$$n \in \mathbb{Z}: \quad [n] : E \longrightarrow E, \quad P \mapsto n \cdot P, \quad E[n] = \frac{\frac{1}{n}\Lambda}{\Lambda} \cong (\mathbb{Z}/n\mathbb{Z})^2$$

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Example: $E/\mathbb{C} : y^2 = x^3 - x$, $(\Lambda = \mathbb{Z} \oplus i\mathbb{Z})$ has the automorphism $(X, Y) \mapsto (-X, i \cdot Y)$, usually denoted as $[i]$ and $[i]^2 = [-1]$.

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No hope to distort a point (over $\mathbb{C}$) if $\mathrm{End}(E) = \mathbb{Z}$. We have to look specifically for the extra endomorphisms/automorphisms in the reduction to $\mathbb{F}_q$.

Endomorphisms over $\overline{\mathbb{F}}_q$

The Frobenius

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Ordinary and Supersingular

$\mathrm{End}(E)$ (over $\overline{\mathbb{F}}_q$) can only be one of the following two:

$$\mathrm{End}(E) = \begin{cases} \text{An order in } \mathbb{Q}(\sqrt{-d}),\ d \in \mathbb{Z}_{>0} \text{ squarefree}, \\ \text{An order in a Quaternion algebra over } \mathbb{Q}, \end{cases}$$

That is a $\mathbb{Q}$-vector space with basis $\{1, i, j, k\}$ such that $ij = k, ji = -k, i^2, j^2 \in \mathbb{Q}$. We say E is *ordinary* in the first case and *supersingular* in the second.

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If $P \in E(\mathbb{F}_q) \cap E[\ell]$ for a prime ℓ, then we evaluate $\phi^2 - t\phi + q \equiv (\phi - 1)(\phi - q) \mod \ell$.

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The Frobenius will distort any point in $E[\ell]$ that is not in an eigenspace!

A Criterion for a distortion map on $E[\ell]$

Let $B = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the endomorphism algebra of $E/\mathbb{F}_q$. Pick an isogeny $\phi \in \mathrm{End}(E) \setminus \mathbb{Z}$, then $K_\phi := \mathbb{Z}[\phi] \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic subfield of B. We have that $\mathbb{Z}[\phi]$ is an order in K_ϕ with discriminant D_ϕ.

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Proposition (D.Charles, FP-team)

If $\ell \mid [\mathrm{End}(E) \cap K_\phi : \mathbb{Z}[\phi]]$, then ϕ is not a distortion map for any $P \in E[\ell] \setminus \{O\}$. Otherwise,

1. $\left(\frac{D_\phi}{\ell}\right) = 1$ if and only if ϕ is a distortion map for all but two cyclic subgroups of $E[\ell]$.
2. $\left(\frac{D_\phi}{\ell}\right) = -1$ if and only if ϕ is a distortion map for $E[\ell]$.
3. $\left(\frac{D_\phi}{\ell}\right) = 0$ if and only if ϕ is a distortion map for all but one cyclic subgroup of $E[\ell]$.

Proof (Sketch)

If $\mathrm{End}(E) \cap K_\phi = \mathbb{Z}[\beta]$ for some endomorphism β, then $\phi = \alpha + f\beta$, for $f = [\mathrm{End}(E) \cap K_\phi : \mathbb{Z}[\phi]]$ and $\alpha \in \mathbb{Z}$. Thus, if $\ell \mid f$

$$\phi|_{E[\ell]} = \alpha|_{E[\ell]} \quad \text{not a distortion map.}$$

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Now if χ_ϕ is the characteristic polynomial of ϕ with discriminant D_ϕ, and $\overline{\chi}_\phi$ is the reduction mod ℓ, we have to consider the cases:

$$\overline{\chi}_\phi \text{ irreducible}, \quad \overline{\chi}_\phi(x) = (x-\alpha)(x-\beta), \quad \overline{\chi}_\phi(x) = (x-\alpha)^2 \pmod{\ell}$$

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with some extra details to be filled in the third case that we have a 1-dimensional subspace. An interesting subcase of this is $\overline{\chi}_\phi(x) = x^2$ when $\ell \mid [\mathcal{O}_{K_\phi} : \mathrm{End}(E) \cap K_\phi]$.

Consequences

Corollary

Let $m > 1$ an integer coprime to $\mathrm{char}\mathbb{F}_q = p$ and $\phi \in \mathrm{End}(E) \setminus \mathbb{Z}$ with $D_\phi = \mathrm{disc}(\mathbb{Z}[\phi])$.

$$\phi \text{ distorts } E[m] \iff$$

$$\phi \text{ distorts } E[\ell], \text{ for all } \ell \mid m \iff$$

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Transfer via isogenies (FP-team)

Consider $E_1, E_2/\mathbb{F}_q$ and an isogeny $\psi : E_1 \to E_2$, with $d := \deg \psi$ and its dual $\hat{\psi} : E_2 \to E_1$, i.e. $\psi\hat{\psi} = [d]$. Then,

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The Ordinary case

Theorem (Verheul)

Let $P \in E(\mathbb{F}_q)$ be a point of order ℓ. If $E[\ell]$ is not defined over $\mathbb{F}_q$, then P does not admit a distortion map.

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$$E[\ell] \cap E(\mathbb{F}_q) = \langle P \rangle, \quad \pi_q(\phi(P)) = \phi(\pi_q(P)) = \phi(P)$$

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Charles' Theorem

Assume $E[\ell] \subseteq E(\mathbb{F}_q)$ (possibly by extending the initial $\mathbb{F}_q$), set $D = \mathrm{disc}(\mathrm{End}(E))$. Then $(D/\ell) = -1, 1, 0$ classifies the existence of distortion maps on the points of $E[\ell]$.

On the shoulders of giants

$\mathbb{Z}_\ell$ denotes the ℓ-adic integers $\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$.

A Theorem of Tate

Let K be a finite field and E/K be an elliptic curve. Fix a prime $\ell \neq \mathrm{char}(K)$ and let $\mathrm{End}_K(T_\ell(E))$ denote the ring of $\mathrm{Gal}(\overline{K}/K)$-equivariant $\mathbb{Z}_\ell$-linear endomorphisms of $T_\ell(E) := \varprojlim E[\ell^n]$. The canonical map

$$\mathrm{End}_K(E) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \quad \longrightarrow \quad \mathrm{End}_K(T_\ell(E))$$

is an isomorphism of rings.

The supersingular case

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Theorem (FP-team)

Let E be a supersingular elliptic curve over a finite field $\mathbb{F}_q$ and let $m > 1$ be an integer coprime to q. Then $E[m]$ admits a distortion map.

Proof (Sketch)

Step 1: Center of $\mathrm{End}(E)$ is $\mathbb{Z}$ for a supersingular elliptic curve. $B = \mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is a central $\mathbb{Q}$-algebra.

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Step 2: Work over $K = \mathbb{F}_{q^k}$ the minimal field of definition of $\mathrm{End}(E)$. Use Step 1 to show that π_{q^k} is in $\mathbb{Z}$. From the Theorem of Tate deduce that $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathrm{End}(T_\ell(E))$ as $(x \mapsto x^{q^k})$ profinitely generates $\mathrm{Gal}(\overline{K}/K)$ and commutes with every element of $\mathrm{End}(E)$.

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Step 4: Use density of $\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell$ to drop "$\otimes\mathbb{Z}_\ell$" and CRT for $m = \prod \ell_i^{n_i}$.

Proof (Sketch)

Step 1: Center of $\mathrm{End}(E)$ is $\mathbb{Z}$ for a supersingular elliptic curve. $B = \mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is a central $\mathbb{Q}$-algebra.

Step 2: Work over $K = \mathbb{F}_{q^k}$ the minimal field of definition of $\mathrm{End}(E)$. Use Step 1 to show that π_{q^k} is in $\mathbb{Z}$. From the Theorem of Tate deduce that $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathrm{End}(T_\ell(E))$ as $(x \mapsto x^{q^k})$ profinitely generates $\mathrm{Gal}(\overline{K}/K)$ and commutes with every element of $\mathrm{End}(E)$.

Step 3: Use the canonical surjection $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{End}_{\mathbb{Z}/\ell^n\mathbb{Z}}(E[\ell^n])$

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Step 5: Find matrix in $M_2(\mathbb{Z}/m\mathbb{Z})$ with no eigenvectors. This is done by picking for each $E[\ell^n]$ the matrix $A = (0\ \overline{\alpha}, 1\ 0)$ for $\overline{\alpha}$ the image of a quadratic non-redisude α modulo ℓ.

Example 1

Let $E/\mathbb{C}$ with equation $y^2 = x^3 + 1$ and complex multiplication by $[\zeta] : (x, y) \mapsto (\zeta x, y)$ for ζ a primitive third root of unity. Based on the arithmetic of $\mathbb{Q}(\zeta)$, and the Deuring reduction theorem, reducting to $p = 2 \mod 3$ yields a supersingular curve, and $p = 1 \mod 3$ an ordinary.

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$E/\mathbb{F}_{13}$ is ordinary and ζ can be $3 \mod 13$, thus the distortion map is simply $(x, y) \mapsto (3x, y)$. By computation $\#E(\mathbb{F}_{13}) = 12$, so we check the primes $\ell = 2, 3$. $E[3]$ is not defined in $\mathbb{F}_{13}$ so no distortion map on $E[3]$. Also $E[2] = \{\mathcal{O}, (4, 0), (10, 0), (12, 0)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It is easy to see all four points are distorted. Observe that $D_{[\zeta]} = -3$ and $(-3/2) = -1$. (In general $(D/2) = -1$ if $D = \pm 3 \mod 8$).

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For the supersingular $E/\mathbb{F}_{11}$, we have to extend to $\mathbb{F}_{11^2}$ to define $[\zeta]$. Nonetheless, again it holds that as $D_{[\zeta]} = -3$ and $\#E(\mathbb{F}_{11}) = 12$. As $(-3/2) = -1$, the map $[\zeta]$ distorts the entire $E[2]$.

Example 2

Let $E/\mathbb{F}_{13}$ be given by $y^2 = x^3 + 11x + 4$. E is ordinary and $E[2]$ is defined over $\mathbb{F}_{13}$. Its conductor $[\mathcal{O}_K : \mathrm{End}(E)]$ is 2 for $K = \mathbb{Q}(\sqrt{-3})$ (by Deuring reduction from $y^2 = x^3 - 3375/121x + 675/121$ over $\mathbb{C}$). So we expect a nilpotent action, recall $\overline{\chi}_\phi(x) = x^2$.

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Extend the field to $\mathbb{F}_{13^2} = \mathbb{F}_{13}[X]/(X^6 + 10X^3 + 11X^2 + 11^X + 2)$, generated by z. Then $P = (4, 3z^5 + 9z^4 + 7z^3 + 12z + 5)$ is in $E[4]$. The group $\langle P \rangle$ can be the kernel of some isogeny $\psi : E \to E'$. A method by Vélu recovers $E' : y^2 = x^3 + 8x + 9$ and the coordinates of the isogeny ψ given $\langle P \rangle$, and is implemented in SageMath. For this example it holds that $j(E') = j(E)$. Thus there is an isomorphism $\gamma : E' \to E$, $(x, y) \mapsto (4x, -5y)$.

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$$\phi(11, 0) = \mathcal{O},\ \phi(6, 0) = (11, 0),\ \phi(9, 0) = (11, 0).$$

Algorithmic Aspects

Assume that for an elliptic curve over $\mathbb{F}_p$, we do not know which curve $/\mathbb{C}$ was used to construct it.

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Assume that for an elliptic curve over $\mathbb{F}_p$, we do not know which curve $/\mathbb{C}$ was used to construct it.

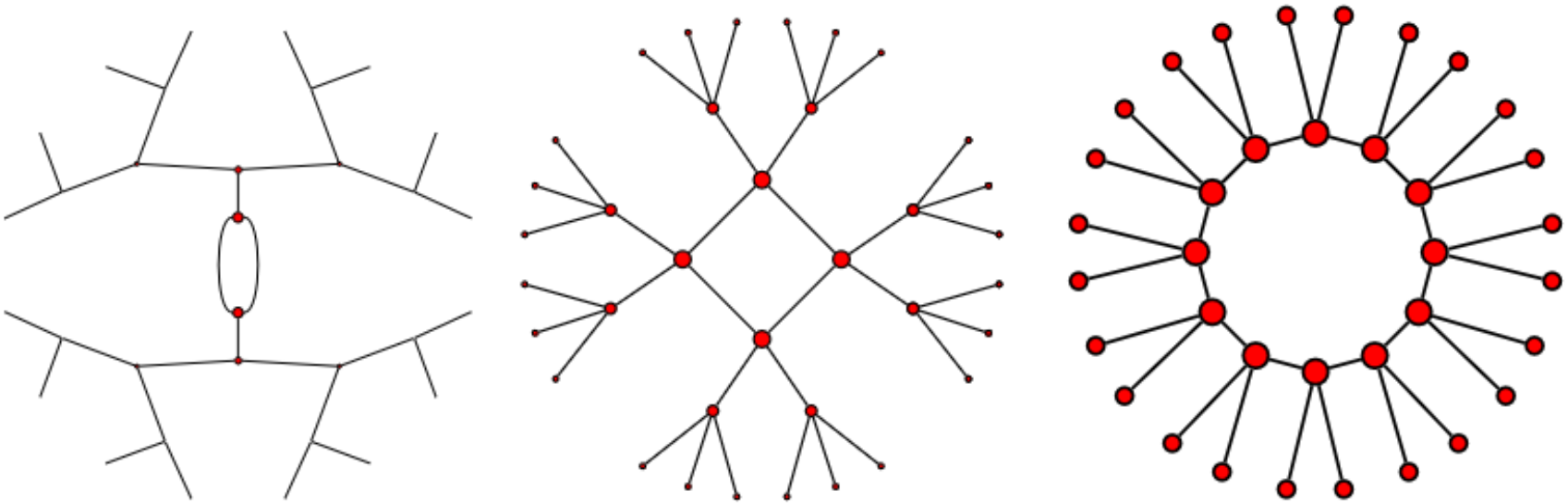
Over $\mathbb{F}_p$ all supersingular curves are isogenous and form a complete isogeny graph. Based on this, Galbraith and Rotger gave an alternative proof of Verheul and Schoof's Theorem by considering isogeny paths on the graphs. Once we reach the same j-invariant as the one we started, we twist with an isomorphism to get a distortion map.

$$E = E_0 \xrightarrow{\psi_1} E_1 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{c-1}} E_{c-1} \xrightarrow{\psi_c} E_c = E' \overset{\gamma}{\simeq} E.$$

Their algorithm is *constructive*.

Algorithmic Aspects

Over $\mathbb{F}_p$ the isogeny graphs form volcanoes:



The same idea is used in the "Distorting the Volcano" paper by M. Fouquet, J. Miret and J. Valera, but there does not exist a deterministic algorithm *yet*.

(Figure from "Ordinary isogeny graphs over $\mathbb{F}_p$: the inverse volcano problem" by H.Bambury, F. Campagna, F. Pazuki)

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Thank you for your attention!